K23P 1410

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.) Examination, October 2023 (2020 Admission Onwards) MATHEMATICS MAT3C13 : Complex Function Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Prove that the sum of the residues of an elliptic function is zero.

2. Define the period module. Show that if f is not a constant function, then the elements of the period module of f are isolated.

3. Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Prove that $[f_1]_b = [g_1]_b$.

4. Show that if G an open connected subset of $\mathbb{C}$, is homeomorphic to the unit disk, then G is simply connected.

5. a) Prove that if $u : G \to \mathbb{C}$ is harmonic, then u is infinitely differentiable.

   b) Define the mean value property.

6. Prove that if $u : G \to \mathbb{R}$ is a continuous function which has the MVP, then u is harmonic.

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Define basis of a period module. Prove that any two bases of the same module are connected by a unimodular transformation.

   b) Prove that an elliptic function without poles is a constant.

8. a) Prove that a non-constant elliptic function has equally many poles as it has zeros.

   b) Prove that zeros $a_1, a_2, \ldots, a_n$ and poles $b_1, b_2, \ldots, b_n$ of an elliptic function satisfy $a_1 + a_2 + \ldots + a_n \equiv b_1 + b_2 + \ldots + b_n \pmod{M}$.

9. a) Prove that for $\operatorname{Re} z > 1$, $\zeta(z)\,\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1}\,dt$.

   b) Define Riemann's functional equation. State and prove Euler's theorem.

Unit – II

10. State and prove Runge's theorem.

11. State and prove Mittag-Leffler's theorem.

12. a) When does a function element (f,D) said to admit unrestricted analytic continuation in G ?

    b) State and prove Monodromy theorem.

Unit – III

13. a) State and prove Jensen's formula. Also state Poisson-Jensen formula.

    b) Suppose $f(0) \neq 0$ in Jensen's formula. Show that if f has a zero at z = 0 of multiplicity m, then $\log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r = -\sum_{k=1}^{n}\log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|d\theta$.

14. a) Define subharmonic and superharmonic function. When does one say that a function satisfies the maximum principle ?

b) Let G be a region and $\phi : G \to \mathbb{R}$ be a continuous function. Then prove that ϕ is subharmonic iff for every region G_1 contained in G and every harmonic function u_1 on G_1, $\phi - u_1$ satisfies the maximum principle on G_1.

c) If ϕ_1 and ϕ_2 are subharmonic functions on G and if $\phi(z) = \max\{\phi_1(z), \phi_2(z)\}$ for each z in G, then show that ϕ is a subharmonic function.

15. Let $D = \{z : |z| < 1\}$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Then prove that there is a continuous function $u : \bar{D} \to \mathbb{R}$ such that

a) $u(z) = f(z)$ for z in ∂D.

b) u is harmonic in D. Also show u is unique and is defined by the formula

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\,dt, \text{ for } 0 \le r < 1, 0 \le \theta \le 2\pi.$$