K24P 0319

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS – Reg./Supple.-(One Time Mercy Chance)/Imp.) Examination, April 2024
(2017 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. Each question carries 4 marks.

1. Let X be a normed space and $A \in BL(X)$. Show that A is invertible if and only if A is bounded below and surjective.

2. Give an example to show that not every bounded sequence in X′ has a weak* convergent subsequence.

3. Let Y be a Banach space, $F_n \in CL(X, Y)$, $F \in BL(X, Y)$ and $||F_n - F|| \to 0$. Prove that $F \in CL(X, Y)$.

4. Let X be an infinite dimensional normed space and $A \in CL(X)$. Prove that $0 \in \sigma_a(A)$.

5. Let H be a Hilbert space. Consider $A, B \in BL(H)$, prove that $(A + B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$.

6. Let H be a Hilbert space and $A \in BL(H)$. Prove that A is normal if and only if $||A(x)|| = ||A^*(x)||$ for all $x \in H$. **(4×4=16)**

P.T.O.

PART – B

Answer **four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. Let $X = l^p$ with the norm $\| \ \|_p$, $1 \le p \le \infty$. For $x = (x(1), x(2), \ldots) \in X$, let $C(x) = (0, x(1), x(2), \ldots)$. Find $\sigma(C)$, $\sigma_e(C)$ and $\sigma_a(C)$.

8. Prove that dual of l^1 is l^∞.

9. a) Let X be a Banach space, $A \in BL(X)$ and $\|A\|^p < 1$ for some positive integer p. Show that $I - A$ is invertible and $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$.

   b) Show that $x_n \xrightarrow{w} x$ in l^1 if and only if $x_n \to x$ in l^1.

Unit – II

10. Let X be a reflexive normed space. Prove that

    a) X is Banach and it remains reflexive in any equivalent norm

    b) X′ is reflexive

    c) Every closed subspace of X is reflexive

    d) X is separable if and only if X′ is separable.

11. a) Let X be a Banach space which is uniformly convex in some equivalent norm. Prove that X is reflexive.

    b) Let X and Y be normed spaces and $F \in BL(X, Y)$. If $F \in CL(X, Y)$, show that $F' \in CL(Y', X')$.

12. Let X be a normed space and $A \in CL(X)$. Prove that every nonzero spectral value of A is an eigenvalue of A.

Unit – III

13. a) Let H be a Hilbert space and $A \in BL(H)$. Prove that $R(A) = H$ if and only if A^* is bounded below.

b) Let H be a Hilbert space and $A \in BL(H)$. Prove that A is unitary if and only if $||A(x)|| = ||x||$ for all $x \in H$ and A is surjective.

c) Give examples of positive operators A and B such that composition operators AB may not be a positive operator.

14. a) State and prove generalized Schwarz inequality.

b) Let $A \in BL(H)$. Prove that $\sigma_e(A) \subset \omega(A)$ and $\sigma(A)$ is contained in closure of $\omega(A)$.

15. a) Let $A \in BL(H)$ be normal. If x_1 and x_2 are eigenvectors of A corresponding to distinct eigenvalues, prove that $x_1 \perp x_2$.

b) Let $A \in BL(H)$ be Hilbert Schmidt operator. Prove that

i) A is compact.

ii) A^* is Hilbert Schmidt operator. **(4×16=64)**