K23P 3297

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS – Supple. (One Time Mercy Chance)/ Imp.) Examination, October 2023
(2017 to 2022 Admissions)
MATHEMATICS
MAT1C03 : Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries 4 marks. **(4×4=16)**

1. Define a neighborhood of a point p. Prove that every neighborhood is an open set.

2. If E is an infinite subset of a compact set K, then prove that E has a limit point in K.

3. When can you say that a function f is said to be differentiable at a point x ? Let f be defined on [a, b]. If f is differentiable at a point $x \in [a, b]$, then prove that f is continuous at x.

4. Suppose f is differentiable in (a, b). If $f'(x) \geq 0$ for all $x \in (a, b)$, then prove that f is monotonically increasing.

5. Give an example of a continuous function, which is not of bounded variation. Justify.

6. Let $f : [a, b] \to R^n$ and $g : [c, d] \to R^n$ be two paths in R^n, each of which is one to one on its domain. Then prove that f and g are equivalent if and only if they have the same graph.

P.T.O.

PART – B

Answer **any four** questions from this part without omitting **any** Unit. **Each** question carries **16** marks. **(4×16=64)**

UNIT – I

7. a) Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, a_2, ..., a_n)$, where $a_k \in A$ (k = 1, 2, ... ,n) and the elements $a_1, a_2, ..., a_n$ need not be distinct. Then prove that B_n is countable.

   b) Let A be the set of all sequences whose elements are the digits 0 and 1. Then prove that this set A is uncountable.

   c) Prove the following :

      i) For any collection $\{G_\alpha\}$ of open sets $\cup_\alpha G_\alpha$ is open.

      ii) For any collection $\{F_\alpha\}$ of closed sets $\text{cap}_\alpha F_\alpha$ is closed.

      iii) For any finite collection $G_1, G_2, ..., G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is open.

      iv) For any finite collection $F_1, F_2, ..., F_n$ of closed sets, $\bigcup_{i=1}^{n} F_i$ is closed.

8. a) Prove that every k-cell is compact.

   b) Let P be a non-empty perfect set in R^k. Then prove that P is uncountable.

9. a) Define uniformly continuous mapping. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then prove that f is uniformly continuous on X.

   b) Let f be monotonic on (a, b). Then prove that the set of points of (a, b) at which f is discontinuous is at most countable.

UNIT – II

10. a) When can you say that a real function f has a local maximum ? Let f be defined on [a, b]; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then prove that $f'(x) = 0$.

    b) State and prove the generalized mean value theorem.

    c) Give an example to show that the mean value theorem fails to be true for complex-valued functions. Justify.

11. a) Define the refinement of a partition P. If P^* is a refinement of P, then prove that $L(P, f, \alpha) \le L(P^*, f, \alpha)$ and $U(P^*, f, \alpha) \le U(P, f, \alpha)$.

b) Prove that $\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$.

12. a) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on [a, b], then prove that

a) $fg \in R(\alpha)$

b) $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

b) Suppose $c_n \ge 0$ for 1, 2, 3, ..., $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. Let f be continuous on [a, b]. Then prove that $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

UNIT – III

13. a) Let $f \in R$ on [a, b]. For $a \le x \le b$, let $F(x) = \int_a^x f(t)\,dt$. Then prove that F is continuous on [a, b]. Also prove that if f is continuous at a point x_0 of [a, b], then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

b) State and prove the fundamental theorem of calculus.

14. a) When can you say that a function f is said to be of Bounded Variation on [a, b] ? If f is monotonic on [a, b], then prove that f is of bounded variation [a, b].

b) If f is continuous on [a, b], and if f′ exists and is bounded in the interior, say $|f'(x)| \le A$ for all x in (a, b) , then prove that f is of bounded variation on [a, b].

c) If f is of bounded variation on [a, b], then prove that f is bounded on [a, b].

15. a) Define Rectifiable paths and its arc length. Consider a path $f : [a, b] \to R_n$ with components $f = (f_1, f_2, ..., f_n)$. Then prove that f is rectifiable if and only if each component f_k is of bounded variation on [a, b]. Also if f is rectifiable, prove that $V_k(a, b) \le \Lambda_f(a, b) \le V_1(a, b) + ... + V_n(a, b)$ (k = 1, 2, ..., n).

b) If f′ is continuous on [a, b], then prove that f is rectifiable and the arc length is $\Lambda_f(a, b) = \int_a^b \|f'(t)\|\,dt$.