K24P 0320

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.S.S. – Reg./Supple. – (One Time Mercy Chance)/Imp.) Examination, April 2024
(2017 Admission Onwards)
MATHEMATICS
MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this part. **Each** question carries **4** marks. **(4×4=16)**

1. Sketch the gradient vector field of the function $f(x_1, x_2) = x_1^2 + x_2^2$.

2. Sketch the graph of the function $f(x_1, x_2) = x_1^2 - x_2^2$.

3. Prove that $X \dot{+} Y = \dot{X} + \dot{Y}$.

4. Define (i) Radius of Curvature (ii) Circle of Curvature, of a plane curve C.

5. Explain why unit speed curves are parametrized by arc length.

6. Find the length of the parametrized curve $\alpha(t) = (\sin t, \cos t, \sin t, \cos t)$ in $[0, 2\pi]$.

PART – B

Answer **four** questions from this part without omitting **any** Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Find the integral curve through (1, –1) of the vector field $X(x_1, x_2) = (x_1, x_2, -x_2, -\frac{1}{2}x_1)$.

   b) Show that the unit n sphere is an n surface in R^{n+1}.

   c) Sketch the typical level curves for c = –1, 0, 1 and graph of the function $f(x_1, x_2) = x_1 - x_2^2$.

P.T.O.

8. a) Prove the following : Lt $S = f^{-1}(c)$ be an n – surface in R^{n+1}, where $f : U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$, an let X be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some t_0 in I, then $\alpha(t) \in S$ for all $t \in I$.

   b) Find the maximum value and minimum value of the function $g(x_1, x_2) = ax_1^2 + bx_2^2 + 2x_1x_2$, $x_1, x_2 \in R$ on the unit circle $x_1^2 + x_2^2 = 1$.

   c) Find the orientations on the cylinder $x_1^2 + x_3^2 = 1$ in R^3.

9. a) Find and sketch the gradient field of the function $f(x_1, x_2) = x_1^2 + x_2^2$.

   b) i) Verify that a surface of revolution is a 2 – surface.

   ii) Sketch the surface of revolution obtained by rotating the curve $x_2 = 2$.

   c) Show that graph of any function $f : R^n \to R$ is a level set for some function $F : R^{n+1} \to R$.

Unit – II

10. a) Prove the following : Let S be an n surface in R^3 and $\alpha : I \to S$ be a geodesic in S with $\dot{\alpha} \neq 0$. Then a vector field X tangent to S along α is parallel along α if and only if both $\|X\|$ and the angle between X and $\dot{\alpha}$ are constant along α.

   b) Compute the Weingarton map for the circular cylinder $x_2^2 + x_3^2 = a^2$ in R^3 $(a \neq 0)$.

11. a) Prove the following :

   i) $\nabla_\upsilon(X + Y) = \nabla_\upsilon(X) + \nabla_\upsilon(Y)$

   ii) $\nabla_\upsilon(fX) = (\nabla_\upsilon f)\, X(p) + f(p)(\nabla_\upsilon X)$

   iii) $\nabla_\upsilon(X.Y) = (\nabla_\upsilon X).Y(p) + X(p).(\nabla_\upsilon Y)$

   b) With the usual notations, prove that $L_p(\upsilon).w = L_p(w).\upsilon$, $\forall \upsilon, w \in S_p$.

   c) With the usual notations, prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot product.

12. a) Find the curvature of the plane curve $C = f^{-1}(0)$ oriented by the outward normal where $f(x_1, x_2) = x_2^2 - x_1$.

   b) Show that i) $D_\upsilon(fX) = (\nabla_\upsilon f)\, X(p) + f(p)D_\upsilon X$

   ii) $\nabla_\upsilon(X.Y) = (D_\upsilon X).Y(p) + X(p).(D_\upsilon Y)$

Unit – III

13. a) Prove the following : Let η be the 1 – form on $R^2 - \{0\}$ defined by

$$\eta = -\frac{x_2}{x_1^2 + x_2^2} + \frac{x_1}{x_1^2 + x_2^2}.$$ Then for $\alpha : [a, b] \to R^2 - \{0\}$ be any closed piece wise smooth parametrized curve in $R^2 - \{0\}$, $\int_\alpha \eta = 2\pi k$.

b) Find the Gaussian curvature of the surface $x_1^2 + x_2^2 - x_3 = 0$ oriented by its outward normal.

14. a) Derive the formula for Gaussian curvature of an oriented n – surface in R^{n+1}.

b) Prove the following : Let S be an n – surface in R^{n+1} and let $p \in S$. Then there exists an open set V about $p \in R^{n+1}$ and a parametrized n surface $\phi : U \to R^{n+1}$ such that ϕ is a one one map from U on to $S \cap V$.

15. a) Obtain a Torus as a parametrized surface in R^3.

b) Prove the following : Let S be an n surface in R^{n+1} and let $f : S \to R^k$. Then f is smooth if and only if $f \circ \phi : U \to R^k$ is smooth for each local parametrization $\phi : U \to S$.

c) Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self adjoint linear transformation on V. Prove that there exist an orthonormal basis for V consisting of eigenvectors of L.